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The fuzzy disc

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Abstract: We introduce a finite dimensional matrix model approximation to the algebra of functions on a disc based on noncommutative geometry. The algebra is a subalgebra of the one characterizing the noncommutative plane with a $*$ product and depends on two parameters N and θ. It is composed of functions which decay exponentially outside a disc. In the limit in which the size of the matrices goes to infinity and the noncommutativity parameter goes to zero the disc becomes sharper. We introduce a laplacian defined on the whole algebra and calculate its eigenvalues. We also calculate the two-points correlation function for a free massless theory (Green's function). In both cases the agreement with the exact result on the disc is very good already for relatively small matrices. This opens up the possibility for the study of field theories on the disc with nonperturbative methods. The model contains edge states, a fact studied in a similar matrix model independently introduced by Balachandran, Gupta and Kürkçüoğlu.

Keywords: Lattice Quantum Field Theory, Chern-Simons Theories, Non-Commutative Geometry, Matrix Models.

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Contents

1. Introduction

Noncommutative Geometry [1]–[4] provides the possibility to describe also ordinary geometries approximating the fields defined on them with finite (matrix) algebras. The archetype of these *fuzzy spaces* is the fuzzy sphere [5]. Along the same lines other fuzzy spaces have been built [6, 7, 8]. There is also the fascinating possibility that the very structure of spacetime at ultrashort length scales might be fuzzy, with a dramatic drop of degrees of freedom at short distances with its consequences on the ultraviolet structure. As additional example matrix models in noncommutative geometry also provide a way [9] to ask questions (for the moment in a toy model) like the number of points or the average dimension of the universe, questions which was unthinkable to ask in the usual context.

The fuzzy approximation is in some sense similar to the usual lattice one, in that a continuous action is substituted by a model with a finite number of degrees of freedom, thus enabling the possibilities to solve field theories with a approximate path integral formalism. In the limit in which matrices become large the approximation improves. However there are some fundamental differences. One advantage of the fuzzy approximation over the usual lattice is that the basic symmetries of the space at hand are preserved, thus fuzzy spheres are SO(3) invariant, fuzzy tori are translationally invariant etc. The original spaces are recovered when the size of the matrices goes to infinity. In this sense noncommutative geometry, and fuzzy spaces in particular, naturally lead to matrix models [10, 11], which can approximate both commutative and noncommutative spaces. The study of field theory on these spaces with numerical methods has started [12, 13, 14] and seems very promising. In particular the authors of [15] solve exactly, on the Moyal plane, a matrix model which has similarities with the one presented in this paper for the disc. The finite matrix

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model can also be useful as a cutoff for renormalization, as advocated for example in [16]. The connections between finite matrix algebras coming from a noncommutative plane and finite geometries has been noted in [17] in the context of noncommutative Chern-Simons theory [18] as a "quantum Hall droplet", while the connection between the projectors we use and the disc appears in [19, 20]. A model which has several points in common with ours has been independently developed by Balachandran, Gupta and Kürkçüoğlu [21] in a matrix model which represents the edge currents of a Chern-Simons theory on an infinite strip. The difference between our model and theirs lies mainly in the way the limit is performed.

In this paper we present a fuzzy approximation to the two-dimensional disc. We hasten to add that although the root of the approximation lies in noncommutative geometry (or even earlier, in quantization) the space of functions being approximated is a normal *commutative* space of functions. In the limit in which matrices become infinite the noncommutativity disappears[1]. Another important aspect of the model is that, while in the lattice approximation the functions, and hence the correlations, can be calculated only at a finite set of points, in the fuzzy approximation presented here approximate functions have values at all points on the plane. We will argue that the matrix model we present corresponds to a theory on a disc in a variety of ways. We start in section 2 with a brief reminder of the noncommutative plane in the form which is useful for our purposes, which makes evident the map between functions on the plane and infinite matrices. In section 3 we introduce the fuzzy disc, a particular finite matrix subalgebra and we argue that the corresponding functions have mainly support on a disc and decrease exponentially outside it. As the size of matrices increases the boundary becomes sharper. We also discuss the form that the ultraviolet-infrared mixing takes in this case and argue the presence of edge states. In order to show that the algebra we are proposing actually corresponds to a disc, in section 4 we introduce derivatives and laplacians and compare the spectrum of the latter in the continuum and in the finite approximation. In section 5 we sketch an approach to field theories on a disc based on the fuzzy approximation. Although the construction is well suited for Montecarlo techniques, the correlation of a free massless scalar field can be solved analytically giving the Green's function in terms of the inverse of a matrix. We calculate it numerically and compare it with the exact case. The final section contains conclusions and outlook. Some technical details of calculations are in the appendix A.

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2. The noncommutative plane as a matrix algebra

We start from functions on $\mathbb{R}^2$, with coordinates x and y. Consider this plane to be like a 'quantum phase space', that is that we "quantize" x and y and give them a nontrivial commutator:

$$[\hat{x}, \hat{y}] = i\frac{\theta}{2}\,. \tag{2.1}$$

[1] In case the field theory under consideration is a nonabelian theory there can of course be other sources of noncommutativity which will remain. It is also possible (depending on the limit taken) to describe theories on noncommutative spaces.

It is convenient to consider the plane as a complex space with $z = x + iy$. The quantized versions of z and $\bar{z}$ are the usual annihilation and creation operators, $a = \hat{x} + i\hat{y}$ and $a^\dagger = \hat{x} - i\hat{y}$ with a slightly unusual normalization, so that their commutation rule is

$$[a, a^\dagger] = \theta\,. \tag{2.2}$$

The parameter θ, which has the dimensions of a square length, plays the role of $\hbar$, but it has no physical meaning, much like the distance between sites in a lattice approximation.

This "quantization" associates operators to functions. There are however ambiguities in this association and in order to give a unambiguous procedure we introduce a map which associates an operator on an Hilbert space to a function on the plane. This is a quantization procedure and it was essentially introduced by Weyl long ago [22]. Given a function on the plane we define a map Ω_θ which to the function $\varphi(z, \bar{z})$ associates the operator $\hat{\varphi}$ defined as follows. Given the function $\varphi(\bar{z}, z)$ consider its Taylor expansion:

$$\varphi(\bar{z}, z) = \sum_{m,n=0}^{\infty} \varphi_{mn}^{\mathrm{Tay}} \bar{z}^m z^n\,, \tag{2.3}$$

to this function we associate the operator

$$\Omega_\theta(\varphi) := \hat{\varphi} = \sum_{m,n=0}^{\infty} \varphi_{mn}^{\mathrm{Tay}} {a^\dagger}^m a^n\,, \tag{2.4}$$

thus we have "quantized" the plane using a normal ordering prescription. To be precise this map is not the one introduced by Weyl (which associates hermitian operators to real functions) but can be cast in the same language since

$$\Omega_\theta(\varphi) = \int \frac{d^2u}{2\pi\theta} \tilde{\varphi}(\bar{u}, u)\, e^{ua^\dagger/\theta} e^{-\bar{u}a/\theta}\,, \tag{2.5}$$

where $\tilde{\varphi}(\bar{u}, u)$ is the Fourier transform of φ.

Because of the similarity of its integral expression to the standard Weyl map we will also refer to it as the weighted Weyl map. The map Ω_θ is invertible. Its inverse is a variation of the Wigner map [23] (the inverse of the Weyl map). It can be efficiently expressed defining the *coherent* states:

$$a\,|z\rangle = z\,|z\rangle\ , \tag{2.6}$$

then it results

$$\Omega_\theta^{-1}(\hat{\varphi}) = \varphi(\bar{z}, z) = \langle z|\,\hat{\varphi}\,|z\rangle\ . \tag{2.7}$$

There is another useful basis on which it is possible to represent the operators. Consider the number operator

$$\mathrm{N} = a^\dagger a\,, \tag{2.8}$$

and its eigenvectors which we indicate[2] by $|n\rangle$:

$$\mathrm{N}\,|n\rangle = n\theta\,|n\rangle\ . \tag{2.9}$$

[2]There is a possible notational confusion here. While it is true that the coherent state for $z = 0$ is the same as the eigenvector of N with zero eigenvalues, it is not true that the coherent state $|z\rangle\,|_{z=n}$ is the same as $|n\rangle$. Since we never consider coherent states with integer values we refrain from the introduction of another symbol for the eigenvectors of N.

We can then express the operators within a density matrix notation:

$$\hat{\varphi} = \sum_{m,n=0}^{\infty} \varphi_{mn} \left|m\right\rangle \left\langle n\right| \,. \tag{2.10}$$

The elements of the density matrix basis have a very simple multiplication rule:

$$\left|m\right\rangle \left\langle n\right| p\rangle \left\langle q\right| = \delta_{np} \left|m\right\rangle \left\langle q\right| \,. \tag{2.11}$$

The connection between the expansions (2.4) and (2.10) is given by:

$$\begin{aligned} a &= \sum_{n=0}^{\infty} \sqrt{(n+1)\,\theta}\, \left|n\right\rangle \left\langle n+1\right| \\ a^{\dagger} &= \sum_{n=0}^{\infty} \sqrt{(n+1)\,\theta}\, \left|n+1\right\rangle \left\langle n\right| \,. \end{aligned} \tag{2.12}$$

Some useful conversion formulas are collected in appendix A. Note that with our conventions x, y, z, a and $a^{\dagger}$ have the dimension of a length, N and θ those of a length square, while density matrices are dimensionless. Applying the dequantization map (2.7) to the operator $\hat{\varphi}$ in the number basis we obtain for the function φ a new expression, analogous to the Taylor expansion (2.3) in terms of the coefficient φ_{nm}, that is

$$\varphi(\bar{z}, z) = e^{-\frac{|z|^2}{\theta}} \sum_{m,n=0}^{\infty} \varphi_{mn} \frac{\bar{z}^m z^n}{\sqrt{n!m!\theta^{m+n}}} \,, \tag{2.13}$$

where we used the expression (A.3). The ordinary Wigner map of these density matrix elements (the Wigner functions) can be found in [24].

The maps Ω_θ and Ω_θ^{-1} yield a procedure of going back and forth from functions to operators. Moreover, the product of operators being noncommutative, a noncommutative $*$ product between functions is implicitly defined as

$$\left(\varphi * \varphi'\right)(\bar{z}, z) = \Omega_\theta^{-1}\left(\Omega_\theta(\varphi)\, \Omega_\theta(\varphi')\right) . \tag{2.14}$$

This product (which is a variation of the Moyal-Grönewold product [25] at the basis of deformation quantization) was first introduced by Voros [26]. Its differential expression is

$$\left(\varphi * \varphi'\right)(\bar{z}, z) = e^{\theta \partial_{\bar{z}'} \partial_{z''}} \varphi(z')\varphi(z'')\Big|_{z=z'=z''} = \varphi\varphi' + \theta \partial_{\bar{z}}\varphi \partial_z \varphi' + O(\theta 2) \,. \tag{2.15}$$

We will indicate the algebra of functions on the plane with this product as $\mathcal{A}_\theta$.

In the density matrix basis, because of (2.11), the product (2.15) simplifies to an infinite row by column matrix multiplication:

$$\left(\varphi * \varphi'\right)_{mn} = \sum_{k=0}^{\infty} \varphi_{mk} \varphi'_{kn} \,. \tag{2.16}$$

Although the map Ω_θ is not defined for $\theta = 0$, using equation (2.15), it is easy to see that, when $\theta \to 0$, the $*$ product goes to the ordinary commutative product. A word of caution, the various maps and products defined here have domains and ranges which are not identical. While the standard Weyl map maps Schwarzian functions into Hilbert Schmidt operators, for the weighted Weyl map (2.4) this is not always the case.

Using relation (A.3) or directly (2.13) it is easy to see that also:

$$\int d^2z\,\varphi(\bar{z},z) = \pi\theta\,\mathrm{Tr}\,\Phi = \pi\theta\sum_{n=0}^{\infty}\varphi_{nn}\,, \tag{2.17}$$

where we have introduced the matrix Φ with components φ_{nm}.

3. The fuzzy disc subalgebra

The main point of the previous section is that the map Ω_θ and its inverse provide a manner to associate to each function, φ, an infinite dimensional matrix Φ. The price to pay is that the commutative product of functions gets deformed through a parameter θ. In this section we define the subalgebras (with respect to the Voros $*$ product) of *finite* $N\times N$ matrices. Considering the functions whose expansion (2.13) terminates when either n or m is larger than a given integer N, it is immediate to see that these functions are closed under $*$ multiplication. In the limit of $N\to\infty$, with θ fixed, the noncommutative plane is recovered.

The subalgebra can be obtained easily from the full algebra of functions via a projection using the idempotent function introduced in a similar context by Pinzul and Stern [20]:

$$P_\theta^N = \sum_{n=0}^{N}\langle z|\,n\rangle\,\langle n|\,z\rangle = \sum_{n=0}^{N}\frac{r^{2n}}{n!\theta^n}e^{-\frac{r^2}{\theta}}\,, \tag{3.1}$$

where we have used the polar decomposition of $z = re^{i\phi}$ and (A.3). The finite sum may be performed thus yielding

$$P_\theta^N = \frac{\Gamma(N+1,\frac{r^2}{\theta})}{\Gamma(N+1)}\,, \tag{3.2}$$

where $\Gamma(n,x)$ denotes the incomplete gamma function. In this notation it is then clear that, in the limit $N\to\infty$ and $\theta\to 0$ with

$$R^2 \equiv N\theta \tag{3.3}$$

fixed, the sum converges to 1 if $r^2/\theta < N$ (namely $r < R$), and converges to 0 otherwise. It has cylindrical symmetry since ϕ does not appear. For N finite the function vanishes exponentially for r larger than R (see figure 1).

Already for $N = 10^3$ it is well approximated (see figure 2) by a step function.

The function P_θ^N is a projector of the algebra of functions on the plane with the $*$ product:

$$P_\theta^N * P_\theta^N = P_\theta^N\,, \tag{3.4}$$

and the subalgebra $\mathcal{A}_\theta^N$ is defined as

$$\mathcal{A}_\theta^N = P_\theta^N * \mathcal{A}_\theta * P_\theta^N\,. \tag{3.5}$$

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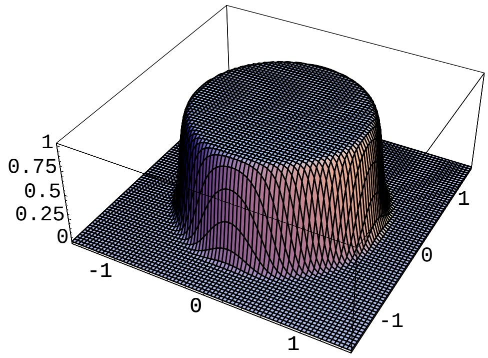

Figure 1: The function P_θ^N for $N = 10^2$.

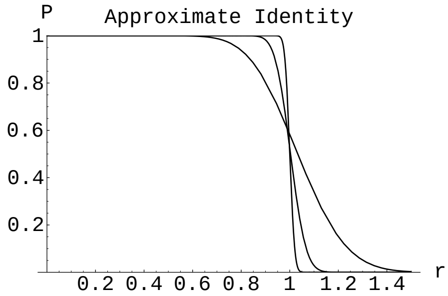

Figure 2: Profile of the spherically symmetric function P_θ^N for the choice $R^2 = N\theta = 1$ and $N = 10, 10^2, 10^3$. As N increases the step becomes sharper.

Cutting at a finite N the expansion provides an infrared cutoff. The cutoff is fuzzy in the sense that functions in the subalgebra are still defined outside the cutoff, but are exponentially damped. For example a gaussian of width θ (corresponding to the element $|0\rangle\langle 0|$) when projected, is mapped into itself, and is nonzero on the whole plane. The map for a normalized gaussian centred around the origin but with width $\alpha > \theta$ also shows the infrared cutoff. Let us investigate the effect of P_θ^N on some functions. We use the notation

$$\Pi_\theta^N(\varphi) \equiv P_\theta^N * \varphi * P_\theta^N = \sum_{m,n=0}^{\infty} \varphi_{mn} e^{-\frac{|z|^2}{\theta}} \frac{\bar{z}^m z^n}{\sqrt{m!n!\theta^{m+n}}} \tag{3.6}$$

and talk of "fuzzy functions" for the projected functions so that $P_\theta^N = \Pi_\theta^N(1)$ is the fuzzy identity. At the level of operators P_θ^N corresponds to the projector

$$\hat{P}_\theta^N = \sum_{n=0}^{N} |n\rangle\langle n| \; . \tag{3.7}$$

We call *fuzzy disc* the space corresponding to the algebra $\mathcal{A}_\theta^N$, which is isomorphic to the algebra of $N+1 \times N+1$ matrices. This, as a matrix algebra, is the same as the one for the fuzzy torus, the fuzzy disc and any fuzzy spaces in general. In [27] the fuzzy sphere is indeed obtained with the use a projector on the Voros algebra of $\mathbb{R}^3$. What makes it a disc are the way to take the correlated limit of θ and N keeping the dimensionful quantity R fixed, which we now pass to describe, and the extra structures we will discuss in section 4.

The limit discussed here must be understood in an heuristic sense, since the algebra of functions on the disc is not approximatively finite one would have to consider it in a weak sense, like the way it is done in [28]. A more rigorous analysis on the line of what is done in [29] would also be desirable. It must also be noted that different ways to take the limit give different commutative or noncommutative spaces, as discussed for example in [30, 19].

We first calculate the effect of the projector on a rotationally symmetric gaussian centred at the origin of width α. Any cylindrically symmetric function $\varphi(r)$ has a simple

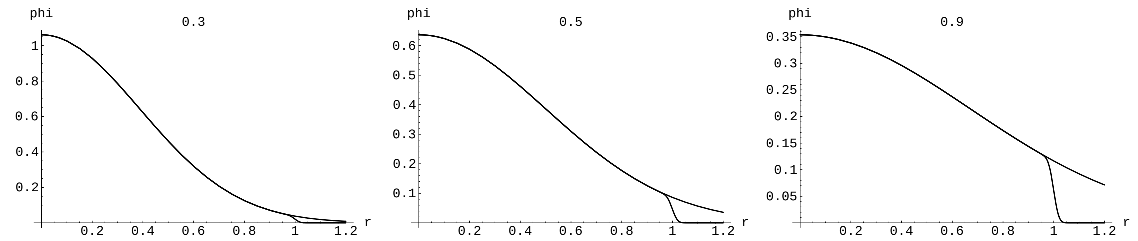

Figure 3: Profile of the spherically symmetric function $\Pi_\theta^N\left(\frac{1}{\pi\alpha}e^{-\frac{r^2}{\alpha}}\right)$ for the choice $R^2 = N\theta = 1$, $N = 10^3$ and for various choices of α compared with the unprojected function. Both functions are plotted, although inside the disc they are practically indistinguishable. The unprojected function is always the larger one.

expansion:

$$\varphi(r) = e^{-\frac{r^2}{\theta}} \sum_{n=0}^{\infty} \varphi_n \frac{r^{2n}}{\theta^n n!}\,, \tag{3.8}$$

where φ_n can be calculated from (A.4). In particular for the normalized gaussian

$$\varphi(r) = \frac{1}{\pi\alpha} e^{-\frac{r^2}{\alpha}} \tag{3.9}$$

we have

$$\varphi_n = \frac{1}{\pi\alpha}\left(1 - \frac{\theta}{\alpha}\right)^n, \tag{3.10}$$

and the series can be summed to give

$$\Pi_\theta^N(\varphi(r)) = e^{-\frac{r^2}{\theta}} \sum_{n=0}^{N} \varphi_n \frac{r^{2n}}{n!\theta^n} = e^{-\frac{r^2}{\alpha}} \frac{\Gamma\left(N+1, r^2\left(\frac{1}{\theta} - \frac{1}{\alpha}\right)\right)}{\pi\alpha\Gamma(N+1)}\,. \tag{3.11}$$

The map Π_θ^N is just an infrared cutoff which in the limit becomes sharper and sharper. Equation (3.11) is plotted in figure 3 for $\theta < \alpha$.

Things are very different if we try to localize the function at a distance smaller than $\sqrt{\theta}$. Figure 4 shows what happens in such a case: for the larger value, $\alpha = .6\theta$, the projected function is undistinguishable from the exact function (both are actually plotted in the figure). For the middle value, $\alpha = .5\theta$, with our numerical approximations, a small "bump" at $r = R$ appears. Already for $\alpha = .49\theta$ the 'bump' has become a large gaussian sitting at the infrared cutoff; the part close to the origin is still there, but it is quickly dwarfed by the infrared bump which grows very fast. For $\alpha \sim .4$ it is already of the order of 10^{17}. Keeping α fixed and increasing N, with R fixed, the bump disappears. The reason for this behaviour is that the factors φ_n in (3.10) become negative and smaller than -1. Therefore the individual terms of the series become larger and larger. Keeping the whole series provides cancellations which do not happen truncating the series at a finite value. In fact the bump becomes negative for N odd. This is a reflection of the fact that, with the weighted Weyl map, the operator corresponding to the Schwarzian function (3.9) with $\alpha \leq \theta/2$ is not compact, and hence it cannot be approximated by finite rank matrices.

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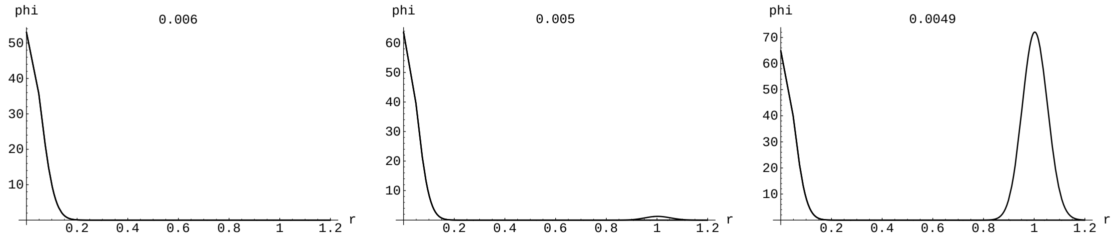

Figure 4: Profile of the spherically symmetric function $\Pi_\theta^N\left(\frac{1}{\pi\alpha}e^{-\frac{r^2}{\alpha}}\right)$ for the choice $R^2 = N\theta = 1$, $N = 10^2$ and for various choices of α.

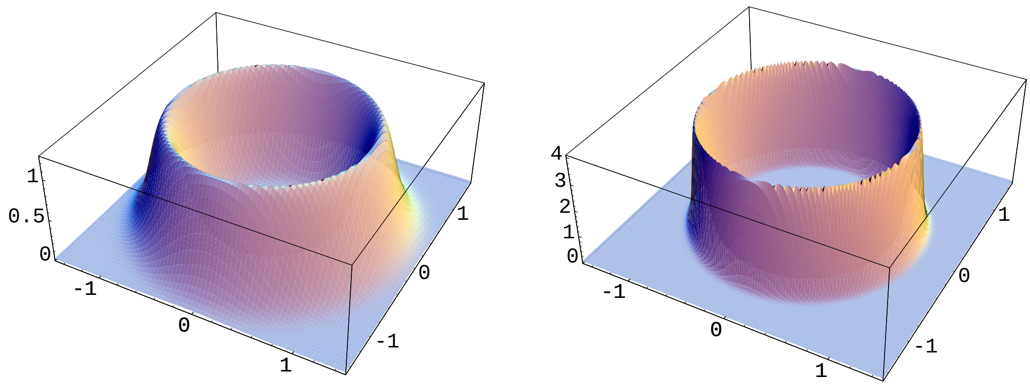

Figure 5: The edge states φ^{edge} for $N = 10$ and $N = 100$.

In general the function φ is close to $\Pi_\theta^N(\varphi)$ if it is mostly supported on the disc of radius R (otherwise it is simply exponentially cut) and if it does not have oscillations of wavelength smaller than $\sqrt{\theta}$. In this case the function becomes very large on the boundary. This is a *compact* example of the *ultraviolet-infrared* mixing [31] which is one of the most interesting characteristics of noncommutative geometry. If we try to localize too much the function, unavoidably an infrared divergence on the boundary of the disc appears.

There are however functions which are localized sharply near the edge of the disc. These are the edge states [32] which play an important role in Chern-Simons theory and have been introduced in these matrix models by A.P. Balachandran, K. Gupta and S. Kürkçüoğlu in [21], and the discussion presented here is inspired by this work. The edge states are simply given by the highest one-dimensional projector:

$$\varphi^{\text{edge}} = \frac{1}{\theta}\langle z|\, N\rangle\,\langle N|\, z\rangle = e^{-\frac{r^2}{\theta}}\frac{r^{2N}}{\theta^{N+1}N!}\,. \tag{3.12}$$

They are plotted in figure 5 for $N = 10$ and $N = 100$. It is evident that as N increases they become sharper.

4. Fuzzy derivatives and fuzzy laplacians

So far we have defined a projection from functions on the plane to a finite dimensional algebra of $N\times N$ matrices and discovered that, with an appropriate choice of N and θ we obtain a good approximation of functions supported on a disc. But, in the spirit of noncommutative geometry, if we want to talk of geometry, we have to define a Dirac operator or a laplacian which give the metric properties of the system [1]. To define derivatives thus we define a pair of operators which act on the same space as the elements of $\mathcal{A}_\theta^N$. Their commutators with a function define what we usually call the derivatives of a function. Of course we require that in the limit these two derivations converge to the usual derivatives of the disc. Combining them with γ matrices would give the Dirac operator. To avoid discussing fermions we will concentrate on the laplacian.

The starting point to define the matrix equivalent of the derivations is:

$$\begin{aligned}\partial_z\varphi &= \frac{1}{\theta}\langle z|\,[a^\dagger,\Omega_\theta(\varphi)]\,|z\rangle\\ \partial_{\bar z}\varphi &= \frac{1}{\theta}\langle z|\,[a,\Omega_\theta(\varphi)]\,|z\rangle\ .\end{aligned}\tag{4.1}$$

The above expression is exact. Acting on an element of the subalgebra $\mathcal{A}_\theta^N$ the derivative takes the functions out of the algebra, a phenomenon not uncommon in noncommutative geometry. However

$$\partial_z(\mathcal{A}_\theta^N)\subset\mathcal{A}_\theta^{N+1}\tag{4.2}$$

analogously for $\partial_{\bar z}$, so that the derivatives of functions of the subalgebra can still be considered finite matrices. Notice that

$$\partial_z\Pi_\theta^N(\varphi)\neq\Pi_\theta^N(\partial_z\varphi)\,,\tag{4.3}$$

the equality obviously holding in the limit. We will use $\partial_z\Pi_\theta^N(\varphi)$ to define the laplacian below.

The fact that we keep a and $a^\dagger$ operators on the full space (hence they are still infinite matrices) is crucial for the identification of the algebra of $N+1\times N+1$ matrices with the approximation to the disc. If we were just to truncate the matrices a and $a^\dagger$, their commutator would not be proportional to the identity. For the fuzzy torus the derivations cannot be expressed as commutators, while for the fuzzy sphere there are three derivations which are not however independent, but are connected by a Casimir. This goes just to say that the same algebra, with different structures, can approximate different spaces.

Rotations are well defined, in fact the generator of rotations on the fuzzy disc is nothing but the number operator N introduced in (2.8) which commutes with $\hat{P}_\theta^N$ just as the generator of rotations on the ordinary disc, the angular momentum operator ∂_ϕ, commutes with P_θ^N. Just as the fuzzy sphere maintains the invariance group of the sphere, the fuzzy disc retains the fundamental symmetry of the disc.

Note that the eigenvalue equations

$$\begin{aligned}\frac{1}{\theta}[a,\varphi] &= \lambda\varphi\\ \frac{1}{\theta}[a^\dagger,\varphi] &= \lambda\varphi\end{aligned}\tag{4.4}$$

have no solution in the space of $N+1\times N+1$ matrices, just as in the commutative case translations on the disc have no eigenvectors. Nevertheless the fuzzy laplacian

$$\hat\nabla^2\hat\varphi := \frac{4}{\theta^2}[a,[a^\dagger,\varphi]] = \frac{4}{\theta^2}[a^\dagger,[a,\varphi]]\tag{4.5}$$

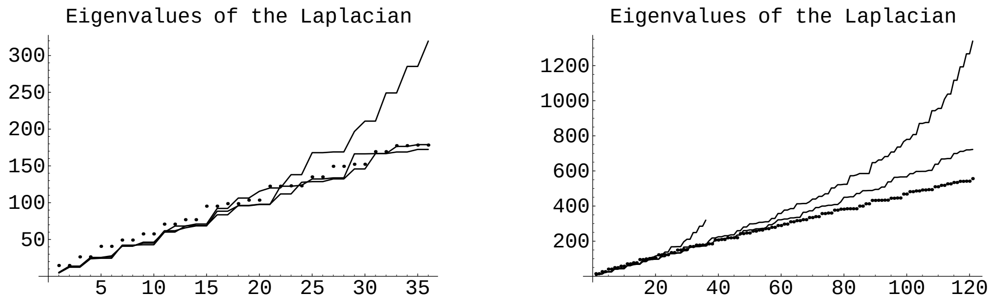

Figure 6: The first eigenvalues of the laplacian on the disc (dots) and the fuzzy laplacian (solid lines) for $N = 5, 10, 15$. The lines corresponding to the three cases can be distinguished because the agreement with the exact case improves as N grows. In the figure on the right the curve which interrupts is the one corresponding to $N = 5$, for which there are only 36 eigenvalues.

can be defined. In particular consider the following matrix model:

$$S = \frac{1}{2\pi\theta} \int d^2z \, \varphi^* * (\nabla^2 \varphi) = \operatorname{Tr} \Phi^\dagger_{mn} \left(\nabla^2\right)_{mnpq} \Phi_{pq} \, , \tag{4.6}$$

where $\left(\nabla^2\right)_{mnpq}$ is implictly defined by (4.5). It is an operator acting on a finite space of dimension $(N+1)^2$, and its eigenvalues can be calculated and compared with the exact commutative case.

As is well known the eigenfunctions of the laplacian on a disc with Dirichlet boundary conditions are the Bessel and the Neumann functions of integer order multiplied by $\exp(im\phi)$. The eigenvalues, E_{mn}, are related to the zeroes of these functions. More precisely, we have

$$E_{mn} = \left(\frac{\text{zeroes}}{R}\right)^2 . \tag{4.7}$$

In figure 6 we show a comparison between the eigenvalues for the exact and approximate laplacians for three values of N. The agreement is fairly good, a fuzzy drum sounds pretty much like a regular drum. Discrepancies start occurring for the $4N^{th}$ eigenvalues, this is to be expected because $4N/R^2 = 4/\theta$ is the energy cutoff of the theory[3]. Most of the eigenvalues (but not all) of the fuzzy laplacian are doubly degenerate, but the unmatched ones become sparser as N increases.

5. Free field theory on the fuzzy disc: Green's functions

The formalism we have developed lends itself readily for matrix approximations to field theories on a disc. For the real scalar case described by the action

$$S = \int d^2z \, \varphi \nabla^2 \varphi + \frac{m^2}{2} \varphi^2 + V(\varphi) \, , \tag{5.1}$$

we consider the fuzzy action

$$S^2_\theta = \frac{1}{\pi} \operatorname{Tr} \Phi \hat{\nabla}^2 \Phi + \frac{m^2}{2} \Phi^2 + V(\Phi) \, . \tag{5.2}$$

[3]We use units for which $\hbar = c = 1$

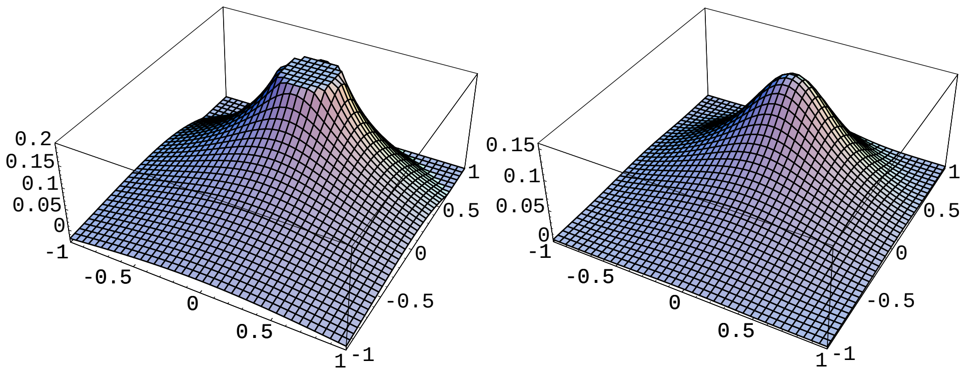

Figure 7: Comparison between the Green's function $G(z, z')$ as function of z for $z' = i/2$. On the left the exact function (with the singularity at $z = z'$ truncated), on the right the approximated function for $N = 20$.

We stress that this is an action entirely of finite dimensional matrices, which can be approached numerically, using for example Montecarlo techniques, a method currently in use for other fuzzy spaces [12, 13, 14]. Here we will content ourselves in calculating the two points Green function for the free massless scalar theory. In this case the path integral may be performed yielding just the inverse of the laplacian with Dirichlet boundary conditions which we have calculated in section 4:

$$\left\langle \varphi(\bar{z}, z)\varphi(\bar{z}', z')\right\rangle = G(z, z') = \langle z| \left(\nabla^2\right)^{-1} \left|z'\right\rangle . \tag{5.3}$$

In the continuous theory on the disc this can be calculated exactly using standard techniques of classical electrodynamics(see for example [33]) yielding

$$G(z, z') = \frac{1}{2\pi} \ln \frac{|z - z'|}{|z'|z| - z|z|^{-1}|} . \tag{5.4}$$

The fuzzy approximation is

$$G_\theta^{(N)}(z, z') = 4 \sum_{m,n,p,q=0}^{N} \frac{e^{-\frac{|z|^2+|z'|^2}{\theta}} (\nabla^{-2})_{mnpq}\bar{z}^p z^q z'^m \bar{z}'^n}{\sqrt{p!q!m!n!\theta^{m+n+p+q}}} . \tag{5.5}$$

This expression can be evaluated numerically. In figures 7 and 8 we show a comparison between the exact Green function on the disc (5.4) and (5.5).

The agreement is, in our opinion, quite remarkable already for a limited number of points. The logarithmic divergence ha been smoothened by the ultraviolet cutoff, but apart from that the two functions are quite similar. The choice of different values of z' gives similar pictures.

6. Conclusions and outlook

We have seen as the fuzzy approximation to the disc works fairly well for the calculations we have attempted, which were based on the structure of the algebra and on the laplacian.

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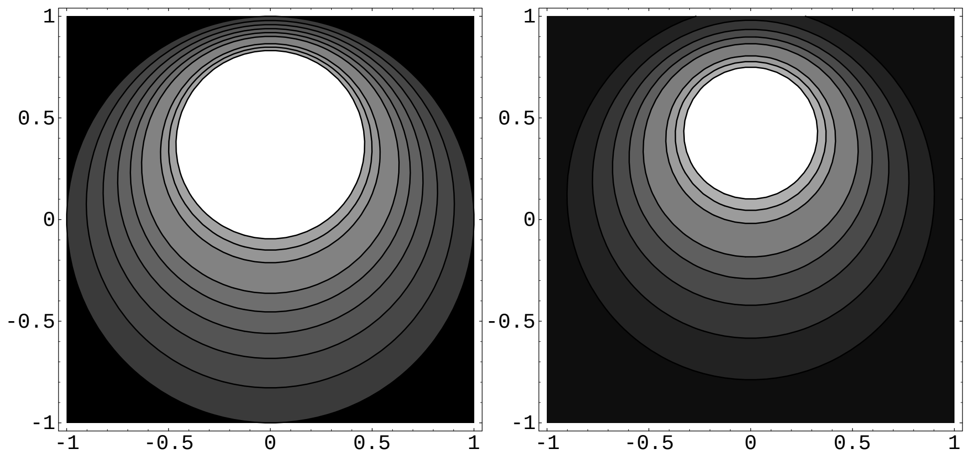

Figure 8: The density plot of the Green's functions with the same parameters as in figure 7. The exact function is on the left.

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The first conclusion that can certainly be drawn is that the fuzzy disc, as a drum, sounds very much like a commutative disc even with an approximation based on relatively small matrices. We performed only one exact analytical calculation as a comparison with the continuum case, but the method can be extended to more involved calculations where it can be a very useful tool. Although the approximation is finite the resulting (fuzzyfied) functions, correlation functions etc. are defined for all points on the disc, and rotational invariance is maintained, thus enabling better comparisons with the continuum theory.

Even though the main stress of this paper has been on the numerical approximations, the fuzzy disc is also an interesting object from the mathematical point of view, and its setting on a more rigorous footing would not only have an intrinsic value, but also help understanding the role of the limits in the approximation of field theory and renormalization.

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A. Conversions between different bases

In this section we collect some useful conversions formulas between the various bases. They are all standard expressions, but care has to be taken because of the nonstandard commutation relation (2.2). The resolutions of the identity are

$$1 = \sum_{n=0}^{\infty} |n\rangle \langle n| = \frac{1}{\pi\theta} \int d^2 z \; |z\rangle \langle z| \; . \tag{A.1}$$

Coherent states are an overcomplete basis and they are not orthogonal, their inner product is:

$$\langle z | \, z' \rangle = e^{-\frac{|z|^2+|z'|^2-2\bar{z}z'}{\theta}} \, , \tag{A.2}$$

while the inner products between a coherent state and an eigenstate of N are given by:

$$\begin{aligned} \langle z | \, n\rangle &= e^{-\frac{|z|^2}{2\theta}} \frac{\bar{z}^n}{\sqrt{n!\theta^n}} \\ \langle n | \, z\rangle &= e^{-\frac{|z|^2}{2\theta}} \frac{z^n}{\sqrt{n!\theta^n}} \, . \end{aligned} \tag{A.3}$$

Known $\varphi^{\rm Tay}_{mn}$ is possible to calculate φ_{mn} from

$$\varphi_{lk} = \sum_{q=0}^{\min(l,k)} \varphi^{\rm Tay}_{l-q \;\, k-q} \frac{\sqrt{l!k!\theta^{l+k}}}{\theta^q q!} \, , \tag{A.4}$$

while the converse is given by

$$\varphi^{\rm Tay}_{mn} = \sum_{p=0}^{\min(n,m)} \frac{(-1)^p \varphi_{m-p \;\, n-p}}{p!\sqrt{(m-p)!(n-p)!\theta^{m+n}}} \, . \tag{A.5}$$

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